

# Sistem linearnih jednačina

Sistem od  $m$  jednačina sa  $n$  nepoznatih zovemo sistem linearnih jednačina

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Sisteme linearnih jednačina možemo rešiti:

a) Gausovom metodom

b) Kramervom metodom (metoda determinanti)

c) Matričnom metodom

d) Kroneker-Kapelijevom metodom

# Kroneker-Kapelijeva metoda

Neka je dat sistem linearnih jednačina  $Ax=b$ , gdje su

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matricu  $\bar{A} = [A \mid b]$  zovemo proširena matrica.

Teorema (Kroneker-Kapeli):

Sistem ima jedinstveno rješenje ako i samo ako je  $\text{rang } A = \text{rang } \bar{A} = n$  ( $n$  broj nepoznatih).

Ako je  $\text{rang } A = \text{rang } \bar{A} < n$  tada sistem ima  $\infty$  mnogo rješenja. ( $n - \text{rang } A$  nepoznatih uzima se proizvoljno)

Ako je  $\text{rang } A < \text{rang } \bar{A}$  tada sistem nema rješenja.

1.) Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$2x + 4y - 5z = -5$$

$$-x - y + z = 0$$

$$2x + y - z = 1$$

$$\text{Rj. } \bar{A} = [A \mid b] = \left[ \begin{array}{ccc|c} 2 & 4 & -5 & -5 \\ -1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{array} \right] \xrightarrow{I_1 \leftrightarrow II_1} \left[ \begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 2 & 4 & -5 & -5 \\ 2 & 1 & -1 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} II_1 + I_1 \cdot 2 \\ III_1 + I_1 \cdot 2 \end{array}} \left[ \begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 2 & -3 & -5 \\ 0 & -1 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{II_1 \leftrightarrow III_1} \left[ \begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -3 & -5 \end{array} \right] \xrightarrow{III_1 + II_1 \cdot 2} \left[ \begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{array} \right]$$

$$\text{rang } A = \text{rang } \bar{A} = 3$$

sistem ima  
jedinstveno  
rješenje

$$-x - y + z = 0$$

$$-y + z = 1$$

$$-z = -3$$

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$$z = 3$$

$$-x - y = -3$$

$$-y = -2$$

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$$y = 2$$

$$-x - 2 = -3$$

$$x = 1$$

Rješenje sistema je uređena trojka  $(1, 2, 3)$ .

2. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 3 \\ 2x_1 + x_2 &= 2. \end{aligned}$$

Rj.  $\bar{A} = [A | b] = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 2 & 1 & 0 & 2 \end{array} \right] \xrightarrow{\substack{II-V \cdot 3 \\ III-V \cdot 2}} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right] \xrightarrow{II \leftrightarrow III} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array} \right]$

$$\xrightarrow{III-II \cdot 2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\text{rang } A = \text{rang } \bar{A} = 2 < 3$

sistem ima  $\infty$  mnogo rješenja

3-2 nepoznatih uzimamo proizvoljno

$x_3 = t$

$-x_2 - 2t = 0$

$x_1 - 2t + t = 1$

$-x_2 - 2x_3 = 0$

$x_2 = -2t$

$x_1 = t + 1$

$x_1 + x_2 + x_3 = 1$

Sistem ima beskonačno mnogo rješenja oblika  $(t+1, -2t, t)$  gdje je  $t \in \mathbb{R}$ .

3. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 6z &= 2 \\ 3x + 6y + 9z &= 5. \end{aligned}$$

Rj.  $\bar{A} = [A | b] = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 5 \end{array} \right] \xrightarrow{\substack{II-V \cdot 2 \\ III-V \cdot 3}} \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array} \right]$

$\text{rang } A = 1, \text{ rang } \bar{A} = 2, \text{ rang } A < \text{rang } \bar{A}$

sistem nema rješenja.

4. Kroneker-Kapelijevom metodom diskutovati rješenja sistema za razne vrijednosti parametra  $\lambda$

$$\begin{aligned} \lambda x + y + z &= 1 \\ x + \lambda y + z &= 2 \\ x + y + \lambda z &= -3 \end{aligned}$$

Rj. za  $\lambda \in (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$  sistem ima jedinstveno rješenje  $\left( \frac{1}{\lambda-1}, \frac{2}{\lambda-1}, \frac{-3}{\lambda-1} \right)$

za  $\lambda = -2$  sistem ima  $\infty$  mnogo rješenja  $\left( \frac{3t-4}{3}, \frac{3t-5}{3}, t \right), t \in \mathbb{R}$

za  $\lambda = 1$  sistem nema rješenja

⊕ Rješiti sistem linearnih jednačina

$$x - y + z = 1$$

$$x - y - z = 2$$

$$x + y - z = 3$$

$$x + y + z = 4$$

Rj. - upute:

Rješimo sistem Kroucher-Kapelijevom metodom

$$\bar{A} = \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \end{array} \right] \sim \dots \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \text{rang}(A) < \text{rang}(\bar{A})$$

Dati sistem nema rješenja

⊕ Rješiti sistem linearnih jednačina

$$x - y + z = 2$$

$$x - y - z = 3$$

$$x + y - z = 4$$

$$x + y + z = 5$$

Rj.-upute:

Rješimo sistem Kruker-Kapelijevom metodom

$$\bar{A} = \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 3 \\ 1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 5 \end{array} \right] \sim \dots \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \text{rang}(A) < \text{rang}(\bar{A})$$

Dati sistem nema rješenja.

⊕ Riješiti sistem jednačina

$$x_1 + 2x_2 - 4x_3 + 8x_4 + 12x_5 = -10$$

$$3x_1 + 7x_2 - 15x_3 + 30x_4 + 45x_5 = -43$$

$$-2x_1 - 3x_2 + 6x_3 - 12x_4 - 18x_5 = 13$$

Rj.-upute:

Sistem demo riješiti Kroneker-Kapelijevom metodom

$$\bar{A} = [A | b] = \left[ \begin{array}{ccccc|c} 1 & 2 & -4 & 8 & 12 & -10 \\ 3 & 7 & -15 & 30 & 45 & -43 \\ -2 & -3 & 6 & -12 & -18 & 13 \end{array} \right] \begin{array}{l} \text{II} + \text{I} \cdot (-3) \\ \text{III} + \text{I} \cdot 2 \end{array}$$

$$\dots \sim \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & -3 & 6 \end{array} \right]^*$$

$$\Rightarrow \text{rang}(A) = 3$$

$$\text{rang}(\bar{A}) = 3$$

$$\text{broj nepoznatih} = 5$$

$\Rightarrow$  sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno

$$\text{npr. } x_4 = s, x_5 = t$$

$$x_1 = 4$$

$$x_2 = 5$$

$$x_3 = 6 + 2s + 3t$$

$$x_4 = s$$

$$x_5 = t$$

$$s, t \in \mathbb{R}$$

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# # Riješiti sistem jednačina

$$\begin{aligned}x_1 + 2x_2 - 4x_3 - 8x_4 - 12x_5 &= -11 \\ -2x_1 - 3x_2 + 5x_3 + 10x_4 + 15x_5 &= 7 \\ -3x_1 - 5x_2 + 10x_3 + 20x_4 + 30x_5 &= 25\end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kruoneker-Kapelijevom metodom

$$\bar{A} = [A | b] = \left[ \begin{array}{ccccc|c} 1 & 2 & -4 & -8 & -12 & -11 \\ -2 & -3 & 5 & 10 & 15 & 7 \\ -3 & -5 & 10 & 20 & 30 & 25 \end{array} \right] \begin{array}{l} \text{II}_v + \text{I}_v \cdot 2 \\ \text{III}_v + \text{I}_v \cdot 3 \end{array}$$

$$\dots \sim \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 2 & 3 & 7 \end{array} \right]$$

$$\Rightarrow \text{rang}(A) = 3$$

$$\text{rang}(\bar{A}) = 3$$

$$\text{broj nepoznatih} = 5$$

}  $\Rightarrow$

sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno  
npr.  $x_4 = s, x_5 = t$

$$x_1 = 5$$

$$x_2 = 6$$

$$x_3 = 7 - 2s - 3t$$

$$x_4 = s$$

$$x_5 = t$$

$$s, t \in \mathbb{R}$$

#) Riješiti sistem jednačina za razne vrijednosti parametra  $\lambda \in \mathbb{R}$ :

$$2x_1 - x_2 + 3x_3 - 7x_4 = 15$$

$$6x_1 - 3x_2 + x_3 - 4x_4 = 7$$

$$4x_1 - 2x_2 + 14x_3 - 31x_4 = \lambda$$

Rj. Rješimo sistem Kramera-Kapelijevom metodom:

$$\bar{C} = [C | b] = \left[ \begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 6 & -3 & 1 & -4 & 7 \\ 4 & -2 & 14 & -31 & \lambda \end{array} \right] \begin{array}{l} \|_V - I_V \cdot 3 \\ \|_V - I_V \cdot 2 \end{array} \left[ \begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 8 & -17 & \lambda - 30 \end{array} \right]$$

$$\|_V + \|_V \left[ \begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 0 & 0 & \lambda - 68 \end{array} \right]$$

1°  $\lambda - 68 \neq 0$   
 $\lambda \neq 68$

$$\text{rang } C = 2$$

$$\text{rang } \bar{C} = 3$$

$\text{rang } C < \text{rang } \bar{C}$  Prema Kramera-Kapelijevoj teoremi sistem nema rješenje

2°  $\lambda - 68 = 0$   
 $\lambda = 68$

$$\text{rang } C = \text{rang } \bar{C} = 2 < 4 \text{ (broj nepoznatih)}$$

Prema Kramera-Kapelijevoj teoremi dvije promjenjive uzimamo proizvoljno, npr.  $x_4 = t, x_1 = s$

$$2x_1 - x_2 + 3x_3 - 7x_4 = 15$$

$$-8x_3 + 17x_4 = -38$$

$$x_4 = t$$

$$-8x_3 + 17t = -38$$

$$-8x_3 = -17t - 38$$

$$x_3 = \frac{17}{8}t + \frac{38}{8} = \frac{17}{8}t + \frac{19}{4}$$

$$x_1 = s$$

$$2s - x_2 + 3\left(\frac{17}{8}t + \frac{38}{8}\right) - 7t = 15$$

$$x_2 = \frac{51t}{8} + \frac{114}{8} + 2s - 7t - 15$$

$$x_2 = -\frac{5}{8}t - \frac{6}{8} + 2s$$

$$x_2 = 2s - \frac{5}{8}t - \frac{3}{4}$$

Za  $\lambda = 68$  rješenje sistema je

$$\left( s, 2s - \frac{5}{8}t - \frac{3}{4}, \frac{17}{8}t + \frac{19}{4}, t \right), t, s \in \mathbb{R}$$



⊕ Riješiti sistem jednačina za razne vrijednosti parametra

$$\lambda \in \mathbb{R}: \begin{cases} 8x_1 + 12x_2 + 7x_3 + \lambda x_4 = 9 \\ 6x_1 + 9x_2 + 5x_3 + 6x_4 = 7 \\ 4x_1 + 6x_2 + 3x_3 + 4x_4 = 5 \\ 2x_1 + 3x_2 + 2x_3 + 2x_4 = 2 \end{cases}$$

Rj. Sistem ćemo rešiti Kroneker-Kapelijevom metodom:

$$\bar{B} = [B | b] = \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 8 & 12 & 7 & \lambda & 9 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 2 & 3 & 2 & 2 & 2 \end{array} \xrightarrow{I_V \leftrightarrow IV_V} \begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ \hline 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 8 & 12 & 7 & \lambda & 9 \end{array} \begin{array}{l} II_V - I_V \cdot 3 \\ III_V - I_V \cdot 2 \\ IV_V - I_V \cdot 4 \end{array}$$

$$\sim \begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ \hline 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & \lambda-8 & 1 \end{array} \begin{array}{l} III_V - II_V \\ IV_V - II_V \end{array} \begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ \hline 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda-8 & 0 \end{array}$$

1° za  $\lambda = 8$  imamo  $\text{rang } B = \text{rang } \bar{B} = 2 < 4$  pa prema Kroneker-Kapelijevoj teoremi sistem ima  $\infty$  mnogo rješenja. Dvije promjenjive uzimamo proizvoljno npr.  $x_1 = t, x_4 = s$

$$\begin{array}{rcl} 2x_1 + 3x_2 + 2x_3 + 2x_4 = 2 & x_3 = -1 & 3x_2 = 4 - 2t - 2s \\ -x_3 + 0x_4 = 1 & 2t + 3x_2 - 2 + 2s = 2 & x_2 = \frac{2}{3}(2 - t - s) \end{array}$$

Rješenje sistema je  $(t, \frac{2}{3}(2-t-s), -1, s)$  gdje su  $s, t \in \mathbb{R}$ .

2° za  $\lambda \neq 8$  imamo  $\text{rang } B = \text{rang } \bar{B} = 3 < 4$  pa prema Kroneker-Kapelijevoj teoremi sistem ima  $\emptyset$  mnogo rješenja. Jednu promjenjivu uzimamo proizvoljno npr.  $x_2 = t$ .

$$\begin{array}{rcl} 2x_1 + 3x_2 + 2x_3 + 2x_4 = 2 & x_4 = 0 & 2x_1 = 4 - 3t \\ -x_3 = 1 & x_3 = -1 & x_1 = 2 - \frac{3}{2}t \\ (\lambda - 8)x_4 = 0 & 2x_1 + 3t - 2 = 2 & \end{array}$$

Rješenje sistema je  $(2 - \frac{3}{2}t, t, -1, 0)$  gdje su  $t \in \mathbb{R}$ .

#) Riješiti sistem jednačina za razne vrijednosti parametra  $\lambda \in \mathbb{R}$ :

$$\begin{aligned} \lambda x_1 - 4x_2 + 9x_3 + 10x_4 &= 11 \\ 2x_1 - x_2 + 3x_3 + 4x_4 &= 5 \\ 4x_1 - 2x_2 + 5x_3 + 6x_4 &= 7 \\ 6x_1 - 3x_2 + 7x_3 + 8x_4 &= 9 \end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijeovom metodom:

$$\bar{A} = [A|b] = \left[ \begin{array}{cccc|c} \lambda & -4 & 9 & 10 & 11 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ 6 & -3 & 7 & 8 & 9 \end{array} \right] \xrightarrow{I_1 \leftrightarrow IV_1} \left[ \begin{array}{cccc|c} 6 & -3 & 7 & 8 & 9 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array} \right] \xrightarrow{II_1 \leftrightarrow I_1}$$

$$\sim \left[ \begin{array}{cccc|c} 2 & -1 & 3 & 4 & 5 \\ 6 & -3 & 7 & 8 & 9 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array} \right] \xrightarrow{II_k \leftrightarrow IV_k} \left[ \begin{array}{cccc|c} x_4 & x_2 & x_3 & x_1 & \\ 4 & -1 & 3 & 2 & 5 \\ 8 & -3 & 7 & 6 & 9 \\ 6 & -2 & 5 & 4 & 7 \\ 10 & -4 & 9 & \lambda & 11 \end{array} \right] \xrightarrow{II_k \leftrightarrow III_k} \left[ \begin{array}{cccc|c} x_2 & x_4 & x_3 & x_1 & \\ -1 & 4 & 3 & 2 & 5 \\ -3 & 8 & 7 & 6 & 9 \\ -2 & 6 & 5 & 4 & 7 \\ -4 & 10 & 9 & \lambda & 11 \end{array} \right]$$

$$\begin{aligned} & \xrightarrow{II_V - I_V \cdot 3} \left[ \begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5 \\ 0 & -4 & -2 & 0 & -6 \\ 0 & -2 & -1 & 0 & -3 \\ 0 & -6 & -3 & \lambda-8 & -9 \end{array} \right] \xrightarrow{II_k \leftrightarrow III_k} \left[ \begin{array}{cccc|c} x_2 & x_1 & x_3 & x_4 & \\ -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array} \right] \xrightarrow{III_V \leftrightarrow II_V} \left[ \begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array} \right] \end{aligned}$$

$$\begin{aligned} & \xrightarrow{III_V - III_V \cdot 2} \left[ \begin{array}{cccc|c} x_2 & x_1 & x_3 & x_4 & \\ -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-8 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

a) Za  $\lambda=8$  imamo  $\text{rang } A = \text{rang } \bar{A} = 2 < 4$  pa prema Kroneker-Kapelijeovom teoremu sistem ima  $\infty$  mnogo rješenja.  
2. promjenjive uzimamo proizvoljno npr.  $x_4 = t \quad x_1 = s$

$$-x_3 - 2x_4 = -3$$

$$-x_2 + 2x_1 + 3x_3 + 4x_4 = 5$$

$$x_3 = 3 - 2t$$

$$-x_2 + 2s + 3(3 - 2t) + 4t = 5$$

$$x_2 = 2s + 9 - 6t + 4t - 5$$

$$x_2 = 2s - 2t + 4$$

Za  $\lambda=8$  rješenje sistema je  $(s, 2s - 2t + 4, 3 - 2t, t)$   
 $s, t \in \mathbb{R}$

b) Za  $\lambda \neq 8$  imamo  $\text{rang } A = \text{rang } \bar{A} = 3 < 4$  pa prema Kroneker-Kapelijeovom teoremu sistem ima  $\infty$  mnogo rješenja.

1. (jednu) promjenjivu uzimamo proizvoljno npr.  $x_4 = t$

$$(\lambda - 8)x_1 = 0$$

$$-x_3 - 2x_4 = -3$$

$$-x_2 + 2x_1 + 3x_3 + 4x_4 = 5$$

Za  $\lambda \neq 8$  rješenje sistema je  $(0, 4 - 2t, 3 - 2t, t)$ .

$$x_1 = 0$$

$$x_3 = 3 - 2t$$

$$-x_2 + 3(3 - 2t) + 4t = 5$$

$$x_2 = 9 - 6t + 4t - 5 = -2t + 4$$

## Cramerovo pravilo (metoda determinanti)

Rješavamo sistem oblika  $A \cdot x = b$  gdje je  $A = [a_{ij}]_{n \times n}$ ,  $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$   
 $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ .  $D_k$  determinanta koja se dobije od  $D$  ( $D = \det A$ ) kada se umjesto  $k$ -te kolone u  $D$  stave slobodni članovi  $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ .

- a) za  $D \neq 0$  sistem ima jedinstveno rješenje  $x = \frac{D_x}{D}$ ,  $y = \frac{D_y}{D}$ ,  $z = \frac{D_z}{D}$   
b) za  $D = 0$ ; ( $D_x \neq 0$  ili  $D_y \neq 0$  ili  $D_z \neq 0$ ) sistem nema nijedno rješenje  
c) za  $D = D_x = D_y = D_z$  ne možemo ništa zaključiti (sistem može imati  $\infty$  mnogo rješenja ili nemati nijedno rješenje) (potrebna su dalja ispitivanja)

Metodom determinanti riješiti sistem jednačina

$$\begin{aligned} 2x - y - z &= 4 \\ 3x + 4y - 2z &= 11 \\ 3x - 2y + 4z &= 11 \end{aligned}$$

$$R_j: D = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{vmatrix} \begin{array}{l} \text{II}_v + \text{I}_v \cdot (-2) \\ \text{III}_v + \text{I}_v \cdot 4 \end{array} \begin{vmatrix} 2 & -1 & -1 \\ -1 & 6 & 0 \\ 11 & -6 & 0 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 6 \\ 11 & -6 \end{vmatrix} = -(-6 - 66) = 60$$

$$D_x = \begin{vmatrix} 4 & -1 & -1 \\ 11 & 4 & -2 \\ 11 & -2 & 4 \end{vmatrix} \begin{array}{l} \text{II}_v - \text{I}_v \cdot 2 \\ \text{III}_v + \text{I}_v \cdot 4 \end{array} \begin{vmatrix} 4 & -1 & -1 \\ 3 & 6 & 0 \\ 27 & -6 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 3 & 6 \\ 27 & -6 \end{vmatrix} = -(-18 - 162) = 180$$

$$D_y = \begin{vmatrix} 2 & 4 & -1 \\ 3 & 11 & -2 \\ 3 & 11 & 4 \end{vmatrix} \begin{array}{l} \text{I}_k + \text{III}_k \cdot 2 \\ \text{II}_k + \text{III}_k \cdot 4 \end{array} \begin{vmatrix} 0 & 0 & -1 \\ -1 & 3 & -2 \\ 11 & 27 & 4 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 3 \\ 11 & 27 \end{vmatrix} = -(-27 - 33) = 60$$

$$D_z = \begin{vmatrix} 2 & -1 & 4 \\ 3 & 4 & 11 \\ 3 & -2 & 11 \end{vmatrix} \begin{array}{l} \text{II}_v + \text{I}_v \cdot 4 \\ \text{III}_v - \text{I}_v \cdot 2 \end{array} \begin{vmatrix} 2 & -1 & 4 \\ 11 & 0 & 27 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 11 & 27 \\ -1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 11 & 9 \\ -1 & 1 \end{vmatrix} = 3(11 + 9) = 60$$

$$x = \frac{D_x}{D} = \frac{180}{60} = 3; \quad y = \frac{D_y}{D} = \frac{60}{60} = 1; \quad z = \frac{D_z}{D} = \frac{60}{60} = 1$$

Rješenje sistema je  $x=3$ ,  $y=1$  i  $z=1$

Metodom determinanti riješiti sistem jednačina:

$$2x + 4y - 5z = -5$$

$$-x - y + z = 0$$

$$2x + y - z = 1$$

$$R_j: x=1, y=2, z=3$$

1) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra  $\lambda$ :

$$(\lambda-2)x - 3y + 2z = 1$$

$$3x - 3y + (\lambda-3)z = 1$$

$$x - y + 2z = -1$$

$$D = \begin{vmatrix} \lambda-2 & -3 & 2 \\ 3 & -3 & \lambda-3 \\ 1 & -1 & 2 \end{vmatrix} \begin{array}{l} I_k + III_k \\ III_k + II_k \cdot 2 \end{array} \begin{vmatrix} \lambda-5 & -3 & -4 \\ 0 & -3 & \lambda-9 \\ 0 & -1 & 0 \end{vmatrix} = (\lambda-5) \begin{vmatrix} -3 & \lambda-9 \\ -1 & 0 \end{vmatrix} = -(\lambda-5)(\lambda-9)$$

$$D_x = \begin{vmatrix} 1 & -3 & 2 \\ 1 & -3 & \lambda-3 \\ -1 & -1 & 2 \end{vmatrix} \begin{array}{l} IV + III \\ II + III \end{array} \begin{vmatrix} 0 & -4 & 4 \\ 0 & -4 & \lambda-1 \\ -1 & -1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} -4 & 4 \\ -4 & \lambda-1 \end{vmatrix} = (-1)(-4) \begin{vmatrix} 1 & 4 \\ 1 & \lambda-1 \end{vmatrix} = 4(\lambda-5)$$

$$D_y = \begin{vmatrix} \lambda-2 & 1 & 2 \\ 3 & 1 & \lambda-3 \\ 1 & -1 & 2 \end{vmatrix} \begin{array}{l} IV + III \\ II + III \end{array} \begin{vmatrix} \lambda-1 & 0 & 4 \\ 4 & 0 & \lambda-1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 4 \\ 4 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 4 = (\lambda-1-4)(\lambda-1+4) = (\lambda-5)(\lambda+3)$$

$$D_z = \begin{vmatrix} \lambda-2 & -3 & 1 \\ 3 & -3 & 1 \\ 1 & -1 & -1 \end{vmatrix} \begin{array}{l} I_k + II_k \\ III_k + II_k \end{array} \begin{vmatrix} \lambda-5 & -3 & 1 \\ 0 & -3 & 1 \\ 0 & -1 & -1 \end{vmatrix} = (\lambda-5) \begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix} = 4(\lambda-5)$$

Diskusija

1°  $\lambda \neq 5$  i  $\lambda \neq 9$  ( $D \neq 0$ ) Sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{4(\lambda-5)}{(\lambda-5)(\lambda-9)} = \frac{4}{\lambda-9}, \quad y = \frac{D_y}{D} = \frac{\lambda+3}{\lambda-9}, \quad z = \frac{D_z}{D} = \frac{4}{\lambda-9}$$

2°  $\lambda = 9$

$D=0, D_x \neq 0 \Rightarrow$  sistem nema rješenja

3°  $\lambda = 5 \Rightarrow D = D_x = D_y = D_z = 0$  na osnovu Cramerovog pravila ne možemo ništa zaključiti. Potrebno je uraditi sistem na drugi način.

za  $\lambda = 5$  sistem postaje

$$3x - 3y + 2z = 1 \quad (1)$$

$$3x - 3y + 2z = 1 \quad (2)$$

$$x - y + 2z = -1 \quad (3)$$

$$(1) = (2)$$

$$(2) - (3): 2x - 2y = 2$$

$$x = y + 1$$

$$x - y + 2z = -1$$

$$y + 1 - y + 2z = -1$$

$$2z = -2$$

$$z = -1$$

sistem ima beskonačno mnogo rješenja, koji su oblika  $(t+1, t, -1), t \in \mathbb{R}$

2) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra  $\lambda$ :

$$(\lambda+4)x + y + z = 2$$

$$x + y + z = \lambda + 5$$

$$3x + 3y + (\lambda+7)z = 3$$

$$D = (\lambda+4)(\lambda+3) \quad t \in \mathbb{R}$$

$$D_x = -(\lambda+4)(\lambda+3) \quad (t, 5-t, -3)$$

$$D_y = (\lambda+3)(\lambda+4)(\lambda+3) \quad (-1, 2-5, 5)$$

$$D_z = -3(\lambda+3)(\lambda+4) \quad s \in \mathbb{R}$$

# Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra  $\lambda$

$$x + y + z = 4$$

$$x + \lambda y + z = 3$$

$$x + 2\lambda y + z = 4$$

Rj. Sistem rješavamo Cramerovom metodom

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 2\lambda & 1 \end{vmatrix} \xrightarrow{\text{II}_V - \text{III}_V} \begin{vmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 1 & 2\lambda & 1 \end{vmatrix} = -\lambda \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 3 & \lambda & 1 \\ 4 & 2\lambda & 1 \end{vmatrix} \xrightarrow{\text{I}_V - \text{II}_V} \begin{vmatrix} 1 & 1-\lambda & 0 \\ 3 & \lambda & 1 \\ 1 & \lambda & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1-\lambda \\ 1 & \lambda \end{vmatrix} = -(\lambda - (1-\lambda)) = 1-\lambda-\lambda = 1-2\lambda$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix} \xrightarrow{\text{III}_k - \text{I}_k} \begin{vmatrix} 1 & 4 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 0$$

$$D_z = \begin{vmatrix} 1 & 1 & 4 \\ 1 & \lambda & 3 \\ 1 & 2\lambda & 4 \end{vmatrix} \xrightarrow{\text{I}_V - \text{II}_V} \begin{vmatrix} 0 & 1-\lambda & 1 \\ 1 & \lambda & 3 \\ 0 & \lambda & 1 \end{vmatrix} \xrightarrow{\text{III}_V - \text{II}_V} \begin{vmatrix} 1-\lambda & 1 \\ \lambda & 1 \end{vmatrix} = -(\lambda - 1 - \lambda) = 2\lambda - 1$$

Kako je  $D=0$  to sistem može da ima beskonačno mnogo rješenja ili da nema rješenja.

1°  $\lambda = \frac{1}{2}$

$$D=0, D_x=0, D_y=0, D_z=0$$

$$x + y + z = 4$$

$$2 - z + y + z = 4$$

$$y = 2$$

Sistem ćemo riješiti Gausovom metodom

$$x + y + z = 4$$

$$x + \frac{1}{2}y + z = 3 \quad / \cdot 2$$

$$x + y + z = 4$$

$$x + y + z = 4 \quad (1)$$

$$2x + y + 2z = 6 \quad (2)$$

$$(2) - (1): x + z = 2$$

$$x = 2 - z$$

Za  $\lambda = \frac{1}{2}$  sistem ima  $\infty$  mnogo rješenja koja su oblika  $(2-t, 2, t)$  gdje je  $t \in \mathbb{R}$ .

2°  $\lambda \neq \frac{1}{2}$

$D=0, D_x \neq 0 \Rightarrow$  sistem za  $\lambda \neq \frac{1}{2}$  nema rješenja

(#) Odrediti vrijednost parametra  $k$  tako da sistem

$$8z - 3x - 6y = kx$$

$$2x + y + 4z = ky$$

$$4x + 3y + z = kz$$

ima beskonačno mnogo rješenja, zatim naci. ta rješenja za najveću dobijenu vrijednost parametra  $k$ .

f.) Nepoznate sa desne strane prebacimo na lijevu i grupirajmo u vrijednosti uz  $x$ ,  $y$  i  $z$ .

$$(-3-k)x - 6y + 8z = 0$$

$$2x + (1-k)y + 4z = 0$$

$$4x + 3y + (1-k)z = 0$$

$$\begin{vmatrix} -3-k & -6 & 8 \\ 2 & 1-k & 4 \\ 4 & 3 & 1-k \end{vmatrix} = 0$$

$$|_k + III_k: \begin{vmatrix} 5-k & -6 & 8 \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

$$|_V - III_V: \begin{vmatrix} 0 & -9 & 7+k \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

Ovo je homogeni sistem linearnih jednačina. Trivijalno rješenje je  $(0,0,0)$ . Sistem ima beskonačno mnogo rješenja ako je  $\Delta = 0$ .

$$(-9) \begin{vmatrix} 1-k & -3 \\ 7+k & 1-k \end{vmatrix} + (5-k) \begin{vmatrix} -9 & 4 \\ -7-k & 1-k \end{vmatrix} = 0$$

$$(-9)(-6k - 30) + (5-k)(-36 - 7 + 6k + k^2) = 0$$

$$\underline{-36k + 18 \cdot 0 + (-45) + 30k + 5k^2 + 43k - 6k^2 - k^3 = 0}$$

$$-k^3 - k^2 + 37k - 35 = 0 \quad | \cdot (-1)$$

$$k^3 + k^2 - 37k + 35 = 0$$

$$k^3 - k^2 + 2k^2 - 2k - 35k + 35 = 0$$

$$k^2(1-k) + 2k(k-1) - 35(k-1) = 0$$

$$(k-1)(k^2 + 2k - 35) = 0$$

$$(k-1)(k+7)(k-5) = 0$$

$$k_1 = 1, k_2 = -7, k_3 = 5$$

Za  $k=5$  imamo.

$$8x + 6y - 8z = 0 \quad \dots (1)$$

$$2x - 4y + 4z = 0 \quad \dots (2)$$

$$4x + 3y - 4z = 0 \quad \dots (3)$$

$$(2) + (3): 6x - y = 0$$

$$\Rightarrow y = 6x$$

$$(2) \rightarrow 2x - 24x + 4z = 0$$

$$\therefore 4z = 22x$$

$$z = \frac{11x}{2}$$

(1)  $\equiv$  (3) jer se (3) dobija dijeljenjem (1) sa 2.

Za  $k=5$  sistem ima rješenja  $(t, 6t, \frac{11t}{2})$  gdje je  $t \in \mathbb{R}$  proizvoljno.

# Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra  $\lambda$ :

$$\begin{aligned} x - y - \lambda z &= 1 \\ (\lambda+1)y + (\lambda-1)z &= 0 \\ (\lambda+1)x - (\lambda+1)z &= 1 \end{aligned}$$

Rj.  $D = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & -(\lambda+1) \end{vmatrix} \stackrel{III_k + I_k}{=} \begin{vmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & 0 \end{vmatrix} = (\lambda+1) \begin{vmatrix} -1 & -(\lambda-1) \\ \lambda+1 & \lambda-1 \end{vmatrix} =$

$$= (\lambda+1)(\lambda-1) \begin{vmatrix} -1 & -1 \\ \lambda+1 & 1 \end{vmatrix} = \lambda(\lambda-1)(\lambda+1)$$

$$D_x = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} \stackrel{III_v - I_v}{=} \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 0 & 1 & -1 \end{vmatrix} \stackrel{-1 + \lambda + 1}{=} \begin{vmatrix} \lambda+1 & \lambda-1 \\ 1 & -1 \end{vmatrix} = \lambda - 1 - \lambda + 1 = -2\lambda$$

$$D_y = \begin{vmatrix} 1 & 1 & -\lambda \\ 0 & 0 & \lambda-1 \\ \lambda+1 & 1 & -(\lambda+1) \end{vmatrix} = -(\lambda-1) \begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -(\lambda-1)(1 - \lambda - 1) = \lambda(\lambda-1)$$

$$D_z = \begin{vmatrix} 1 & -1 & 1 \\ 0 & \lambda+1 & 0 \\ \lambda+1 & 0 & 1 \end{vmatrix} = (\lambda+1) \begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -\lambda(\lambda+1)$$

$D=0$  ako  $\lambda=0$ , ili  $\lambda=1$ , ili  $\lambda=-1$

Diskusija

1°  $\lambda \neq 0$ ;  $\lambda \neq 1$ ;  $\lambda \neq -1$  sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{-2\lambda}{\lambda(\lambda-1)(\lambda+1)} = \frac{-2}{(\lambda-1)(\lambda+1)}, \quad y = \frac{D_y}{D} = \frac{1}{\lambda+1}, \quad z = \frac{D_z}{D} = \frac{-1}{\lambda+1}$$

2°  $\lambda=1$ ,  $D=0$ ,  $D_x \neq 0 \Rightarrow$  sistem nema rješenja

3°  $\lambda=-1$ ,  $D=0$ ,  $D_x \neq 0 \Rightarrow$  sistem nema rješenja

4°  $\lambda=0$ ,  $D=D_x=D_y=D_z=0$  iz ovoga ne možemo ništa zaključiti

Za  $\lambda=0$  sistem postaje

$$\begin{aligned} x - y &= 1 & (1) \\ y - z &= 0 & (2) \\ x - z &= 1 & (3) \end{aligned}$$

(1):  $x - y = 1$

(2)-(3):  $-x + y = -1$   
 $x = y + 1$

$x - z = 1$   
 $-z = -(y+1) + 1$

$-z = -y$   
 $z = y$  Sistem ima  $\infty$  mnogo rješenja  $(t+1, t, t)$ ,  $t \in \mathbb{R}$

#) riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra  $a$ :

$$\begin{aligned} x + y - z &= 0 \\ x - y + az &= 1 \\ -x - 3y + (a+2)z &= a^2 \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & a \\ -1 & -3 & a+2 \end{vmatrix} \begin{array}{l} \underline{\underline{I_k + III_k}} \\ \underline{\underline{II_k + III_k}} \end{array} \begin{vmatrix} 0 & 0 & -1 \\ a+1 & a-1 & a \\ a+1 & a-1 & a+2 \end{vmatrix} = (-1) \begin{vmatrix} a+1 & a-1 \\ a+1 & a-1 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & a \\ a^2 & -3 & a+2 \end{vmatrix} \begin{array}{l} \underline{\underline{II_k + III_k}} \\ \underline{\underline{III_k + III_k}} \end{array} \begin{vmatrix} 0 & 0 & -1 \\ 1 & a-1 & a \\ a^2 & a-1 & a+2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & a-1 \\ a^2 & a-1 \end{vmatrix} = (-1)(a-1) \begin{vmatrix} 1 & 1 \\ a^2 & 1 \end{vmatrix}$$

$$\begin{aligned} D_y &= \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & a \\ -1 & a^2 & a+2 \end{vmatrix} \begin{array}{l} \underline{\underline{I_k + III_k}} \\ \underline{\underline{II_k + III_k}} \end{array} \begin{vmatrix} 0 & 0 & -1 \\ a+1 & 1 & a \\ a+1 & a^2 & a+2 \end{vmatrix} = (-1) \begin{vmatrix} a+1 & 1 \\ a+1 & a^2 \end{vmatrix} = (-1)(a+1) \begin{vmatrix} 1 & 1 \\ 1 & a^2 \end{vmatrix} = (-1)(a+1)(a^2-1) \\ &= (-1)(a+1)(1-a^2) = (a-1)(a^2-1) \\ &= (a-1)^2(a+1) \\ D_z &= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & -3 & a^2 \end{vmatrix} \begin{array}{l} \underline{\underline{I_k - II_k}} \\ \underline{\underline{II_k + III_k}} \end{array} \begin{vmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & -3 & a^2 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 1 \\ 2 & a^2 \end{vmatrix} = (-1)(2a^2-2) = (-2)(a+1)(a-1) \\ &= (-1)(a-1)(a+1)^2 \end{aligned}$$

Diskusija

$$D=0 \quad \forall a \in \mathbb{R}$$

$$1^\circ a \neq 1 \quad ; \quad a \neq -1$$

$D=0$  ;  $D_x \neq 0$  sistem nema rješenja

$$2^\circ a = 1$$

$D=D_x=D_y=D_z=0$ , sistem postaje

$$\begin{aligned} x + y - z &= 0 & (I) \\ x - y + z &= 1 & (II) \\ -x - 3y + 3z &= 1 & (III) \end{aligned}$$

$$\begin{aligned} (I) + (II): & -2y + 2z = 1 \\ (II) + (III): & -4y + 4z = 2 \\ \hline & 2z = 2y + 1 \\ & z = y + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} x &= z - y \\ x &= \frac{y}{2} \end{aligned}$$

Sistem ima  $\infty$  mnogo rješenja oblika  $(\frac{1}{2}, t, t + \frac{1}{2})$  gdje je  $t \in \mathbb{R}$ .

$$\begin{aligned} (I) + (III): & -2y = 1 \\ (II) + (III): & -4y = 2 \end{aligned}$$

$$3^\circ a = -1$$

$D=D_x=D_y=D_z=0$ , sistem postaje

$$\begin{aligned} x + y - z &= 0 & (I) \\ x - y - z &= 1 & (II) \\ -x - 3y + z &= 1 & (III) \end{aligned}$$

$$\begin{aligned} y &= -\frac{1}{2} \\ (I) + (II): & 2x - 2z = 1 \\ (III) - 3 \cdot (II): & -4x + 4z = 2 \\ \hline & 2x = 2z + 1 \\ & x = z + \frac{1}{2} \end{aligned}$$

Sistem ima  $\infty$  mnogo rješenja oblika  $(t + \frac{1}{2}, -\frac{1}{2}, t)$ ,  $t \in \mathbb{R}$



#) Diskutovati rješenja sistema u zavisnosti od parametra  $\lambda$ :

$$2x - \lambda y + 2z = 1$$

$$x + y + 2z = 0$$

$$-x + (-\lambda - 3)y - 4z = \lambda$$

Rj. Sistem ćemo riješiti Cramerovim pravilima.

$$D = \begin{vmatrix} 2 & -\lambda & 2 \\ 1 & 1 & 2 \\ -1 & -\lambda-3 & -4 \end{vmatrix} \begin{array}{l} I_k - II_k \\ III_k - II_k \cdot 2 \end{array} \begin{vmatrix} 2+\lambda & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 2\lambda+2 \\ \lambda+2 & 2\lambda+2 \end{vmatrix} = (\lambda+2) \begin{vmatrix} 1 & 2\lambda+2 \\ 1 & 2\lambda+2 \end{vmatrix}$$

$$D_x = \begin{vmatrix} 1 & -\lambda & 2 \\ 0 & 1 & 2 \\ \lambda & -\lambda-3 & -4 \end{vmatrix} \begin{array}{l} III_k - II_k \cdot 2 \\ III_k - I_k \cdot 2 \end{array} \begin{vmatrix} 1 & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} 1 & 2\lambda+2 \\ \lambda & 2\lambda+2 \end{vmatrix} = (2\lambda+2) \begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ -1 & \lambda & -4 \end{vmatrix} \begin{array}{l} III_k - I_k \cdot 2 \\ III_k - I_k \cdot 2 \end{array} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ -1 & \lambda & -2 \end{vmatrix} = (2\lambda+2)(1-\lambda) = 2(\lambda+1)(1-\lambda) = (-1) \begin{vmatrix} 1 & -2 \\ \lambda & -2 \end{vmatrix} = (-1)(-2) \begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = 2(1-\lambda)$$

$$D_z = \begin{vmatrix} 2 & -\lambda & 1 \\ 1 & 1 & 0 \\ -1 & -\lambda-3 & \lambda \end{vmatrix} \begin{array}{l} I_k - II_k \\ III_k - I_k \cdot 2 \end{array} \begin{vmatrix} 2+\lambda & -\lambda & 1 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 \\ \lambda+2 & \lambda \end{vmatrix} = (\lambda+2) \begin{vmatrix} 1 & 1 \\ 1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda-1)$$

Diskusija:

$$D=0, D_x=2(1+\lambda)(1-\lambda), D_y=2(1-\lambda), D_z=(\lambda+2)(\lambda-1)$$

1°  $\lambda \neq -1$ ;  $\lambda \neq 1$ ;  $\lambda \neq -2$

imamo  $D=0$ ;  $D_x \neq 0$  sistem nema rješenja

2°  $\lambda = -2$  imamo  $D=0$ ;  $D_x \neq 0$  sistem nema rješenja

3°  $\lambda = -1$  imamo  $D=0$ ,  $D_x=0$ ,  $D_y \neq 0$  sistem nema rješenja

4°  $\lambda = 1$  imamo  $D=D_x=D_y=D_z=0$  sistem je potrebno ispitati na drugi način.

Za  $\lambda=1$  sistem postaje

$$8x - 4y + 8z = 4 \quad (1)$$

$$4x + 4y + 8z = 0 \quad (2)$$

$$-x - 4y - 4z = 1 \quad (3)$$

$$(1)+(2): 12x + 16z = 4$$

$$(3)+(2): 3x + 4z = 1$$

$$3x = 1 - 4z$$

$$x = \frac{1-4z}{3}$$

$$y = -x - 2z$$

$$y = \frac{4z-1}{3} - \frac{6z}{3}$$

Sistem ima  $\infty$  mnogo rješenja, oblika  $(\frac{1-4t}{3}, \frac{2t-1}{3}, t)$   $t \in \mathbb{R}$

$$2x - y + 2z = 1 \quad | \cdot 4$$

$$x + y + 2z = 0 \quad | \cdot 4$$

$$-x - 4y - 4z = 1$$

#) riješiti sistem jednačina i diskutovati rješenje u zavisnosti od parametra

$$\begin{aligned}x + y + bz &= 1 - b \\x - by - z &= 2 \\bx - y + z &= 2b\end{aligned}$$

Rješavamo sistem Cramerovom metodom

$$D = \begin{vmatrix} 1 & 1 & b \\ 1 & -b & -1 \\ b & -1 & 1 \end{vmatrix} \xrightarrow{I_k + III_k} \begin{vmatrix} b+1 & 1 & b \\ 0 & -b & -1 \\ b+1 & -1 & 1 \end{vmatrix} = (b+1) \begin{vmatrix} 1 & 1 & b \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} \xrightarrow{I_v - III_v}$$

$$= (b+1) \begin{vmatrix} 0 & 2 & b-1 \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} = (b+1) \begin{vmatrix} 2 & b-1 \\ -b & -1 \end{vmatrix} = (b+1) \left[ \begin{matrix} b^2 - b - 2 \\ -2 + (b^2 - b) \end{matrix} \right] =$$

$$= (b+1)(b+1)(b-2)$$

$$D_x = \begin{vmatrix} 1-b & 1 & b \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} \xrightarrow{I_v + III_v} \begin{vmatrix} b+1 & 0 & b+1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} = (b+1) \begin{vmatrix} 1 & 0 & 1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} =$$

$$\xrightarrow{I_k - III_k} (b+1) \begin{vmatrix} 0 & 0 & 1 \\ 3 & -b & -1 \\ 2b-1 & -1 & 1 \end{vmatrix} = (b+1) \begin{vmatrix} 3 & -b \\ 2b-1 & -1 \end{vmatrix} = (b+1) \frac{2b^2 - b - 3}{0 = 1 + 2b = 25} =$$

$$= (b+1) \cdot 2 \left(b - \frac{3}{2}\right) (b+1) \quad \frac{15}{4}$$

$$D_y = \begin{vmatrix} 1 & 1-b & b \\ 1 & 2 & -1 \\ b & 2b & 1 \end{vmatrix} \xrightarrow{I_k + III_k} \begin{vmatrix} b+1 & 1-b & b \\ 0 & 2 & -1 \\ b+1 & 2b & 1 \end{vmatrix} = (b+1) \begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 1 & 2b & 1 \end{vmatrix} \xrightarrow{III_v - I_v}$$

$$= (b+1) \begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 0 & 3b-1 & 1-b \end{vmatrix} = (b+1) \begin{vmatrix} 2 & -1 \\ 3b-1 & 1-b \end{vmatrix} = (b+1) (2 - 2b + 3b - 1) =$$

$$= (b+1)(b+1)$$

$$D_z = \begin{vmatrix} 1 & 1 & 1-b \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \xrightarrow{I_v + III_v} \begin{vmatrix} b+1 & 0 & b+1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} = (b+1) \begin{vmatrix} 1 & 0 & 1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \xrightarrow{I_k - III_k}$$

$$= (b+1) \begin{vmatrix} 0 & 0 & 1 \\ -1 & -b & 2 \\ -b & -1 & 2b \end{vmatrix} = (b+1) \begin{vmatrix} -1 & -b \\ -b & -1 \end{vmatrix} = (b+1) (1 - b^2) = -(b+1)(b^2 - 1)$$

$$= -(b+1)(b-1)(b+1)$$

Diskusija: a)  $D \neq 0$  tj.  $b \neq -1$ ;  $b \neq 2$

sistem ima jednostrano rješenje  $x = \frac{D_x}{D} = \frac{(2b-3)(b+1)^2}{(b+1)^2(b-2)} = \frac{2b-3}{b-2}$

$$y = \frac{D_y}{D} = \frac{(b+1)^2}{(b+1)^2(b-2)} = \frac{1}{b-2} \quad ; \quad z = \frac{D_z}{D} = \frac{-(b-1)(b+1)^2}{(b-2)(b+1)^2} = -\frac{b-1}{b-2}$$

b)  $b = -1 \Rightarrow D = D_x = D_y = D_z = 0$  sistem trebamo riješiti drugim načinom

Za  $b = -1$  sistem postaje

$$\begin{array}{r} x + y - z = 2 \\ x + y - z = 2 \\ \hline -x - y + z = -2 \quad | \cdot (-1) \end{array}$$

Sve tri jednačine su iste  $\Rightarrow$  Sistem ima  $\infty$  mnogo rješenja. Ako uzmemo  $x = t, y = s$  rješenja sistema su  $(t, s, t + s - 2)$  ← dvije promjenjive uzimamo proizvoljno

c)  $b = 2 \Rightarrow D = 0, D_x = 9 \neq 0 \Rightarrow$   
Sistem za  $b = 2$  nema rješenja

# Homogeni sistemi linearnih jednačina

Homogeni sistem linearnih jednačina je oblika  $A \cdot x = 0$

gdje je

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Teorema: Homogeni sistem ima netrivialna rješenja ako je  $D=0$  ( $\det A=0$ ).

1) Riješiti homogeni sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 & (1) \\ 3x_1 + x_2 - x_3 &= 0 & (2) \\ 2x_1 + x_2 &= 0 & (3) \end{aligned}$$

Rj. (1)+(2)

$$\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \quad | \cdot 2 \\ \hline 4x_1 + 2x_2 &= 0 \\ 4x_1 + 2x_2 &= 0 \end{aligned}$$

$$4x_1 + 2x_2 = 0 \quad | :2$$

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ \text{sistem ima } \infty \text{ mnogo rješenja} \\ x_2 &= -2x_1 \\ x_1 = t, \quad x_2 = -2t, \\ t \in \mathbb{R} \end{aligned}$$

Sistem ima beskonačno mnogo rješenja oblika  $(t, -2t, t)$

2) Naći  $\lambda$  tako da sistem

$$\begin{aligned} 3x + y + \lambda z &= 0 \\ 4x - 8y + \lambda z &= 0 \\ 5x - 3y + 3z &= 0 \end{aligned}$$

ima netrivialna rješenja pa naći rješenja.

Rj.

$$D = \begin{vmatrix} 3 & 1 & \lambda \\ 4 & -8 & \lambda \\ 5 & -3 & 3 \end{vmatrix} \begin{array}{l} ||v+lv \cdot 8 \\ ||v+lv \cdot 3 \end{array} \begin{vmatrix} 3 & 1 & \lambda \\ 28 & 0 & 9\lambda \\ 14 & 0 & 3\lambda+3 \end{vmatrix} = - \begin{vmatrix} 28 & 9\lambda \\ 14 & 3\lambda+3 \end{vmatrix} = (-14) \cdot 3 \begin{vmatrix} 2 & 3\lambda \\ 1 & \lambda+1 \end{vmatrix} = -42(-\lambda+2)$$

Za  $\lambda=2$  ( $D=0$ ) u sistemu postoje netrivialna rješenja.

Sistem sad izgleda:

$$\begin{aligned} 3x + y + 2z &= 0 & | \cdot 3 & & 9x + 3y + 6z &= 0 & (1) \\ 4x - 8y + 2z &= 0 & | \cdot 3 & & 12x - 24y + 6z &= 0 & (2) \\ 5x - 3y + 3z &= 0 & | \cdot 2 & & 10x - 6y + 6z &= 0 & (3) \end{aligned}$$

$$\begin{aligned} (3)-(1): \quad x - 9y &= 0 \\ (2)-(1) \quad 3x - 27y &= 0 \quad | :3 \\ x - 9y &= 0 \end{aligned}$$

$x = 9y, \quad z = -14y$  postoji  $\infty$  mnogo rješenja  
 $(9t, t, -14t), t \in \mathbb{R}$   
su rješenja sistema

3) Za koje vrijednosti  $\lambda$  sistem ima netrivialna rješenja

$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + \lambda x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + \lambda x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + \lambda x_4 &= 0 \end{aligned}$$

Rj. za  $\lambda=1$  ili  $\lambda=-3$